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Chern–Simons Action on a Finite Point Space

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We apply Connes' noncommutative geometry to a finite point space. The explicit Chern–Simons action on this finite point space is obtained.

KEY WORDS: Chern–Simons action; Fredholm module; finite point space.

1. INTRODUCTION

Discrete spaces and corresponding physical theories have been discussed extentively in the literature. See, for example, Bombelli *et al.* (1987), Feynman (1982), Finkelstein (1969), Minsky (1982), Ruark (1931), Snyder (1947), 't Hooft (1990), and Yamamoto (1984, 1985). In the framework of Connes' noncommutative geometry (Connes, 1985, 1994), finite spaces have been considered to build models in particle physics (Chamseddine *et al.*, 1993; Chamseddine and Connes, 1996, 1997; Connes, 1990, 1995, 1996; Connes and Lott, 1990; Coquereaux *et al.*, 1991; Kastler, 1993, 1996; Varilly and Gracia-Bondia, 1993). Differential calculus and gauge theories on finite spaces or finite groups were proposed in Cammarata and Coquereaux (1995), Dimakis and Müller-Hoissen (1994a,b), Krajewski (19980, Paschke and Sitarz (1996), Sitarz (1992, 1995) and references therein. The explicit actions of gauge fields on finite point spaces were obtained in Hu (2000) and Hu and Sant'Anna (2002, 2003).

In this paper we apply Connes' noncommutative geometry to a finite point space. The explicit Chern–Simons action on this finite point space is obtained.

2. DIFFERENTIAL CALCULUS ON A *n*-POINT SPACE

We briefly review the differential calculus on a n-point space. More detailed account of the construction can be found in Cammarata and Coquereaux (1995),

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Dimakis and Müller-Hoissen (1994a,b), Hu (2000), and Hu and Sant'Anna (2002, 2003).

Let M be a space of n points $i_1, \ldots, i_n$ $(n < \infty)$, and $\mathcal{A}$ the algebra of complex functions on M with $(fg)(i) = f(i)g(i)$. Let $p_i \in \mathcal{A}$ defined by

$$p_i(j) = \delta_{ij}. \tag{1}$$

It follows that p_i is a projector in $\mathcal{A}$ $(i = 1, \ldots, n)$. Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i)p_i,$$

where $f(i) \in \mathbf{C}$, a complex number. The algebra $\mathcal{A}$ can be extended to a universal differential algebra $\Omega(\mathcal{A}) = \oplus_{r=0}^{\infty}\Omega^r(\mathcal{A})$ (where $\Omega^0(\mathcal{A}) = \mathcal{A}$) via the action of a linear operator $d : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A})$ satifying

$$d1 = 0, \quad d^2 = 0, \quad d(\omega_r\omega') = (d\omega_r)\omega' + (-1)^r\omega_r d\omega',$$

where $\omega_r \in \Omega^r(\mathcal{A})$. 1 is the unit in $\Omega(\mathcal{A})$.

From the above properties, the set of projectors p_i satisfy the following relations:

$$p_i dp_j = -(dp_i)p_j + \delta_{ij}dp_i, \tag{2}$$

$$\sum_i dp_i = 0. \tag{3}$$

$\Omega(\mathcal{A})$ is an involutive algebra given by

$$(a_0da_1\cdots da_n)^* = da_n^*\cdots da_1^*a_0^*,$$

where $a_0, a_1, \ldots, a_n \in \mathcal{A}$.

The universal first order differential calculus Ω^1 is generated by $p_idp_j (i \neq j)$, $i, j = 1, 2, \ldots, n$. Notice that p_idp_i is the linear combinations of $p_idp_j (i \neq j)$. The compositions of $p_idp_j (i \neq j)$, $i, j = 1, 2, \ldots, n$, generate the higher order universal differential calculus on M.

Let $\mathcal{E} = \mathcal{A}^m$ be a free $\mathcal{A}$-module. A connection on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that

$$\nabla(\Psi a) = (\nabla\Psi)a + \Psi \otimes da, \tag{4}$$

for all $\Psi \in \mathcal{E}$, $a \in \mathcal{A}$.

Any connection on $\mathcal{E}$ is of the form $\nabla = d + A$ with $A^* = -A$. A is called a connection 1-form. We can regard A as an element of $M_m(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. Here $M_m(\mathcal{A})$ is a $m \times m$ matrix algebra over $\mathcal{A}$. A can be written as $A = \sum_{i,j} A_{ij}p_idp_j$ with $A_{ij} \in M_m(\mathbf{C})$, a $m \times m$ complex matrix, and $A_{ii} = 0$, a $m \times m$ zero matrix. Especially, $A^* = -\sum_{i,j} A_{ji}^*p_idp_j$. From $A^* = -A$, we have

$$A_{ij}^* = A_{ji}. \tag{5}$$

Notice that the structure of A_{ij} is independent of the universal differential algebra $\Omega(\mathcal{A})$ over M. This means

$$A = \sum_{i,j} A_{ij} p_i dp_j = \sum_{i,j} p_i dp_j A_{ij}. \tag{6}$$

Let $\mathsf{G} \subset \text{End}_{\mathcal{A}}(\mathcal{E}) = M_m(\mathcal{A})$ be a gauge group of $\mathcal{E}$. Then $\mathsf{G} = \sum_i \mathsf{G}_i p_i$ with $\mathsf{G}_i \subset M_m(\mathbf{C})$. Notice that

$$\mathsf{G}_1 = \mathsf{G}_2 = \cdots = \mathsf{G}_n = G. \tag{7}$$

There is a natural action of G on the space of connections given by

$$\nabla' = g\nabla g^{-1} : \Psi \mapsto g\nabla(g^{-1}\Psi),$$

with $\Psi \in \mathcal{E}$ and $g \in \mathsf{G}$. The connection 1-form A satisfies

$$A' = gAg^{-1} + gdg^{-1}. \tag{8}$$

Here $g = \sum_i g_i p_i \in \mathsf{G}$, and $g_i \in \mathsf{G}_i = G$.

The curvature of ∇ is defined by $F = \nabla^2$. It follows that

$$F = dA + A^2. \tag{9}$$

F transforms in the usual way, $F' = gFg^{-1}$. From $(dA)^* = -dA^* = dA$ and $(A^2)^* = A^2$, one has $F^* = F$.

The Chern–Simons action on M reads

$$S = \text{Tr}\, AdA + \frac{2}{3}\text{Tr} A^3. \tag{10}$$

3. FROM FREDHOLM MODULE TO CHERN–SIMONS ACTION ON *M*

One of the basic ideas in Connes' noncommutative differential geometry is the Fredholm module (Connes, 1994, and references therein). Applying the Fredholm module to the universal algebra $\Omega(\mathcal{A})$ discussed in the previous section, one can obtain an explicit Chern–Simons action on the finite space M.

The Fredholm module $(\mathcal{A}, \mathcal{H}, D)$ is composed as the following (Hu, 2000; Hu and Sant'Anna, 2002, 2003): $\mathcal{A}$ is the algebra on M defined in the previous section. $\mathcal{H}$ is a n-dimensional linear space over the complex field $\mathbf{C}$, i.e., $\mathcal{H}$ is just the direct sum $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$, $\mathcal{H}_i = \mathbf{C}$. The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\pi(f) = \begin{pmatrix} f(1) & 0 & \dots & 0 \\ 0 & f(2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(n) \end{pmatrix},$$

with $f \in \mathcal{A}$. D is a Hermitian $n \times n$ matrix with $D_{ij} = \overline{D}_{ji}$, and D_{ij} is a linear mapping from $\mathcal{H}_j$ to $\mathcal{H}_i$. The following equality defines an involutive representation of $\Omega(\mathcal{A})$ in $\mathcal{H}$,

$$\pi(da) = [D, \pi(a)], \tag{11}$$

where $a \in \mathcal{A}$. To ensure the differential d satisfies

$$d^2 = 0, \tag{12}$$

one has to impose the following condition on D,

$$D^2 = \mu^2 I, \tag{13}$$

where μ is a real constant and I is the $n \times n$ identity matrix. Since the diagonal elements of D commute exactly with the action of $\mathcal{A}$, we can ignore the diagonal elements of D, i.e.,

$$D_{ii} = 0. \tag{14}$$

The projector p_i can be expressed as a $n \times n$ matrix,

$$(\pi(p_i))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}. \tag{15}$$

From Eqs. (11) and (15), it follows that

$$(\pi(p_i dp_j))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j} D_{ij}. \tag{16}$$

Connection matrix. The connection matrix H on M is given by

$$H_{ij} = D_{ij}(A_{ij} + \mathbf{1}). \tag{17}$$

Here $\mathbf{1}$ is the identity in the gauge group G, where G is defind in Eq. (7). One can find that H_{ij} is a $m \times m$ complex matrix with $H^*_{ij} = H_{ji}$. This means that $H = (H_{ij})$ is a $n \times n$ Hermitian matrix with its elements $m \times m$ submatrices. The diagonal elements of H satisfy

$$H_{ii} = 0. \tag{18}$$

From (8) and (17), the transformation rule of H_{ij} reads

$$H'_{ij} = g_i H_{ij} g_j^{-1}. \tag{19}$$

Hermitian matrix $\mathbf{D}$. The Hermitian matrix $\mathbf{D}$ is defined by

$$\mathbf{D} = (\mathbf{D}_{ij}) = (D_{ij}\mathbf{1}). \tag{20}$$

From (13), one has

$$\mathbf{D}^2 = \mu^2 \mathbf{I}, \tag{21}$$

where $\mathbf{I} = (\mathbf{I}_{ij}) = (\delta_{ij}\mathbf{1})$.

Curvature matrix. Applying the Fredholm module to the Eq. (9), one has

$$\pi(dA) = [\mathbf{D}, H - \mathbf{D}]_+ = \mathbf{D}(H - \mathbf{D}) + (H - \mathbf{D})\mathbf{D},$$
$$\pi(A^2) = (H - \mathbf{D})^2.$$

It follows that

$$\pi(F) = H^2 - \mu^2\mathbf{I}. \tag{22}$$

$\pi(F)$ is called the curvature matrix on M. The transformation rule of $\pi(F_{ij})$ satisfies

$$\pi(F'_{ij}) = g_i \pi(F_{ij}) g_j^{-1}. \tag{23}$$

Bianchi identity. From $\pi(dF + AF - FA) = [H, \pi(F)] = [H, H^2 - \mu^2\mathbf{I}] = 0$, one has the Bianchi identity on the finite space M

$$[H, \pi(F)] = 0. \tag{24}$$

Theorem. *The Chern–Simons action on a finite point space M takes the form $S = \frac{2}{3}\mathrm{Tr}H^3$. Here H is the connection matrix on M. S is invariant under the gauge transformation* (19).

Proof: By making use of the Fredholm module, one has the following formulae,

$$\pi(AdA) = (H - \mathbf{D})[\mathbf{D}, H - \mathbf{D}]_+,$$
$$\mathrm{Tr}\pi(AdA) = 2\mathrm{Tr}(\mathbf{D}H^2),$$
$$\pi(A^3) = (H - \mathbf{D})^3,$$
$$\frac{2}{3}\mathrm{Tr}\pi(A^3) = \frac{2}{3}\mathrm{Tr}H^3 - 2\mathrm{Tr}(\mathbf{D}H^2).$$

The action S reads

$$S = \mathrm{Tr}\pi(AdA) + \frac{2}{3}\mathrm{Tr}\pi(A^3) = \frac{2}{3}\mathrm{Tr}H^3. \tag{25}$$

It is obviously that S is unchanged under the gauge transformation (19). □

Example. The $U(1)$ Chern–Simons action on M also takes the form

$$S = \frac{2}{3}\mathrm{Tr}H^3, \tag{26}$$

where H_{ij} is a complex number ($i, j = 1, \ldots, n$).

4. DISCUSSION

By applying Connes' noncommutative geometry to a finite point space. We have obtained the explicit Chern–Simons action on this finite point space. It suggests that the cubic interaction in quantum field theory may have its geometric origin.

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